MATH 207(c)–Analysis IV

1. Solve the initial value problem

$$\begin{cases} \dfrac{d^4u}{dt^4} + \dfrac{d^2u}{dt^2} = e^{-t}, \\ u(0) = \dfrac{du}{dt}(0) = \dfrac{d^2u}{dt^2}(0) = \dfrac{d^3u}{dt^3}(0) = 0. \end{cases}$$

2. Consider the complex function

$$F(z) = \frac{z+1}{z^4 + 5z^2 + 4}.$$

(a) Compute the inverse Laplace transform of F using the integral formula (you can also verify that you got the correct result by alternatively computing the inverse Laplace transform using the properties of $\mathcal{L}[\cdot]$ and Laplace transforms of known examples).

(b) Restricting F on $\mathbb{R}$ (i.e. think of F as a function from $\mathbb{R}$ to $\mathbb{C}$), compute the inverse Fourier transform of F.

3. For the functions $f : [0,1] \to \mathbb{R}$ given below, compute its expansion into a Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n\mathbb{N}\setminus\{0\}} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right).$$

(a) $f(x) = \sin^2(2\pi x)$,

(b) $f(x) = x\sin(2\pi x)$,

(c) $f(x) = e^{-x}$.

4. Consider the initial value problem for the modified heat equation on $x \in [0,1]$ for some $a \in \mathbb{R}$ with Dirichlet conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) - au(x,t) = e^{-2x} \quad \text{for } \ t > 0,\ x \in (0,1), \\ u(x,0) = 0, \\ u(0,t) = u(1,t) = 0 \quad \text{for } \ t > 0. \end{cases}$$

Find an expression for the solution u.

5. Let us consider the same initial value problem as in Exercise 4 but with Neumann boundary conditions instead (in the case of the heat equation, these model an *insulated* endpoint):

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) - au(x,t) = e^{-2x} \quad \text{for } t>0,\, x\in(0,1), \\ u(x,0)=0, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 \quad \text{for } t>0. \end{cases}$$

Find an expression for the solution u. *(Hint: You might want to extend your functions as* even *2-periodic functions in x before decomposing in Fourier modes.)*

6 (Extra). The n-moment of a function $f:[0,+\infty)\to\mathbb{C}$ is defined by

$$\mu_n = \int_0^{+\infty} t^n f(t)\,dt,$$

provided, of course, this integral converges. Show that, if all n-moments of f converge and

$$\sup_{n\in\mathbb{N}} \int_0^{+\infty} t^n |f(t)|\,dt < +\infty,$$

then

$$\mathcal{L}[f](z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!}\mu_n z^n.$$

In particular, it follows form the above that $\mathcal{L}[f](z)$ in this case extends holomorphically at $z=0$ and the moments of f can be calculated in terms of the coefficients of the Taylor expansion of $\mathcal{L}[f](z)$ at $z=0$.